

The Optimal Management of Orchards

(Work in Progress)

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Abstract

A fundamental issue in perennial crop economics is finding the optimal time to replace trees in an orchard. Orchards have two key characteristics: they consist of trees of multiple vintages, and the trees have a non-monotonic yield curve. We present the first analysis of optimal tree replacement in an orchard that has both characteristics. Our results show that cyclical production is optimal in the long-run, and that optimally managed orchards converge to the long-run cycle surprisingly quickly. Our results have implications for orchard valuation, orchard planting, and orchard conversion. We are also the first to provide comparative statics on the long-run cycle radius. In addition, forest management is a natural point of comparison to orchard management. We contrast the optimal replacement rules for orchards and forests to show the qualitative differences.

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1 Introduction

Perennial crops are plants that can be harvested multiple times before replanting. Fruit-bearing trees and vines, such as almonds, olives, apples, and grapes, are prominent examples. From an economic perspective, perennials are durable assets that provide a flow of benefits (‘fruit’) over their productive lifespan. As they approach the end of their lives, they become less productive, leading the grower to consider replacing them with a more productive asset. In this paper we examine the optimal management of an orchard, that is, a perennial crop growing enterprise (which could be a small-holder cocoa grower in Ghana or a large-scale nut grower in California). We explore the trade-off between maximizing the discounted net income from the orchard and the year-to-year variation in that income stream.

Perennial crops are valuable. In the US, the gross returns for fruit and nut production was around 30 billion dollars in 2016, led by grapes, apples, strawberries, and oranges.¹ In the same year, tree nut gross returns were around 10 billion dollars, led by almonds, walnuts, and pistachios (USDA, 2016). Perennial crops are often used as cash crops in developing countries, e.g. coffee and cocoa [XXX Citation needed]. Perennials can provide environmental benefits as well. Perennial grasses such as sugarcane or *miscanthus* are used as a feedstock in the production of low-carbon biofuels (Crago et al., 2010), and early research into perennial grain crops suggests that they can provide substantial environmental benefits relative to their annual cousins, through increasing soil carbon, reducing erosion, and requiring fewer chemical inputs (Glover et al., 2010).

A fundamental issue in perennial crop economics is finding the optimal age at which to replace trees in an orchard. This is a normative issue, relating to what the grower should do to maximize the value of the enterprise. This paper addresses the following broad questions. At what age should a grower replace the trees to maximize the value of the orchard? Should the grower replace some or all of the tree? What will be the steady-state of the optimally managed orchard in the long run? How long will it take for the orchard to reach this state? Our specific research question is: for a

¹Grapes, apples, and oranges are perennials. Strawberries can be grown as an annual or perennial. They are grown as an annual crop in California, accounting for XXX percent of US production (California Strawberry Commission, 2017).

grower with a preference for production smoothing, what is the optimal way to manage an orchard with non-monotonic yields and multiple age classes?

1.1 Non-monotonic yields

The yield of a perennial is typically non-monotonic, or ‘hump-shaped’, over its lifespan. The generic life cycle of a perennial crop is described by Ward and Faris (1968, p.1): “Normally the following production cycle can be expected. (1) No yield for the first few years, (2) yields rapidly increasing to a maximum, (3) yields constant or slowly declining, and (4) yield decreasing (at a faster rate).” This relationship between age and yield, unique to each crop, is the crop’s *yield curve*.

Grounded in the biology of the crop, non-monotonic yield curves are commonly used in generic discussions of the economics of perennials (Tisdell and De Silva, 1986; Mitra et al., 1991). They also used in studies of specific crops, such as cocoa (Mahrizal et al., 2014), almonds (Klonsky et al., 2016), blueberries (Safley et al., 2006), and sugarcane (Margarido and Santos, 2012).

1.2 Multiple age classes

When deciding whether to replace a tree in the orchard, the grower compares the value of keeping the existing tree for another year to replacing it with the best alternative asset. As stated by Faris (1960, p.761), “the optimum time to replace is when the marginal net revenue from the present enterprise is equal to the highest amortized present value of anticipated net revenues from the enterprise immediately following”. The standard [XXX correct language? Cites?] approach to perennial replacement assumes that the current enterprise is a single tree, or a stand of identically-aged trees,² and that the following enterprise is also a single tree. The management issue is thus deciding when to replace the current tree with the next one.

On the other hand, orchard growers typically have trees of multiple age-classes being grown simultaneously, particularly when the grower’s entire enterprise is considered [XXX Add cites to Folley (1969), and Upton and Casey (1974)]. In California, for example, there is a roughly uniform distribution of almond tree ages from 1 to 20 years, with the density of ages declining after 20 years

²A stand of identically-aged trees is conceptually identical to a single tree [XXX True? What about tree loss? Incorporatable into the yield curve...]

(CDFA, 2018). The age-structure—the mixture of trees of different ages in the orchard—affects the productivity of the orchard, due to each tree’s hump-shaped yield curve. The year-to-year variation in the age-structure will influence the year-to-year variation in the orchard’s productivity.

1.3 Production smoothing preferences

If the grower, for whatever reason, has a preference for production smoothing, the year-to-year variation in the productivity of the orchard influences the value of the enterprise to the grower. Such a preference may arise from a variety of sources.

[XXX Need a definition of production smoothing?]

From the perspective of the orchard grower as a single household, the presence of credit constraints correlates production to consumption. A credit-constrained household with a preference of consumption smoothing would therefore also have a preference for production smoothing (Mahrizal et al., 2014).

From a wider, firm-level perspective, a preference for production smoothing may arise due to: capacity constraints, where the fruit of the orchard is processed at a local processing plant (Upton and Casey, 1974; Margarido and Santos, 2012); supply contracts, where the grower has agreed to sell a pre-agreed quantity of fruit each year; or convex replanting costs, which would lead the grower to smooth replanting providing an (implicit) preference for production smoothing.

[XXX This needs more thought...] Thinking of the ‘grower’ as an entire industry, it may have preferences for smoothing production as a method of maintaining profits.

In our analysis, we remain agnostic on the mechanisms behind the preference for production smoothing, using a strictly concave, twice continuously differentiable net benefit function to model the preference. Whatever its cause, the presence of a production smoothing preference implies that that replacement decisions for each tree in the orchard depend on the total production of the orchard, and analyzing replacement decisions at the level of a single tree may be misleading.

2 Replacing a single tree

2.1 The yield curve

The yield curve describes how the yield of the plant changes over its life cycle. We use a deterministic yield curve, abstracting from the fact that observed yield curves in empirical applications will be stochastic functions of multiple variables, including rainfall, temperature profile throughout the growing season, soil type, inputs applied (e.g. fertilizer, pesticide), hours of labor, etc.

We can specify a generic yield curve with four integers representing the non-bearing period, the period of increasing yield, the period of constant yield, and the period of decreasing yields (Mitra et al., 1991). That is, integers P , Q , R , and N , with $0 \leq P < Q \leq R \leq N$, such that

$$f_P \leq f_Q = \dots = f_R \geq \dots \geq f_N$$

and at least one strict inequality between f_0 and f_Q . That is, the yield curve is monotonically increasing, plateaus, then monotonically decreases until the end of the tree's lifespan. P corresponds to the end of the non-bearing period, Q corresponds with the beginning of the yield plateau, R with the end of the yield plateau, and N with the end of the tree's lifespan. A stylized yield curve is shown in figure 1.

They require only monotonic increasing and decreasing for the increasing and decreasing phases of the AY relationship. There could be arbitrarily many inflection points. There could also be other flat regions, and their model allows an initial non-bearing period.

In this paper we focus on two special cases of their general relationship: a two-age class model (young trees and old trees), where old trees are less productive than young trees; and a three-age-class model (young, medium, and old trees) where the old age-class may be either more or less productive than medium trees.

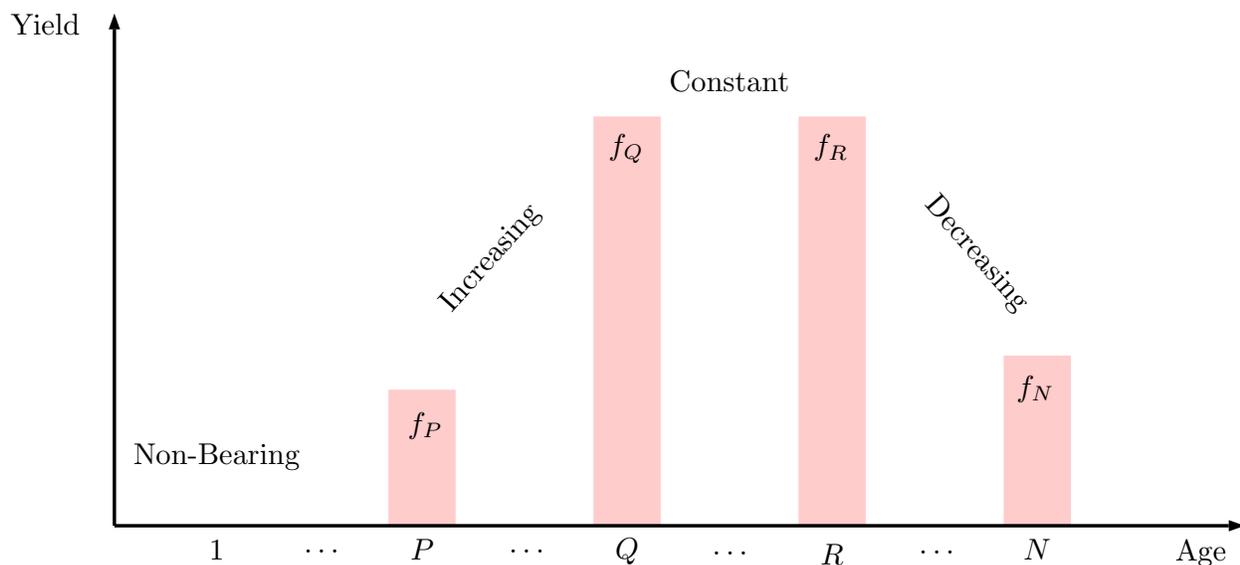


Figure 1: A generic non-monotonic (hump-shaped) yield curve. It is not necessary for a perennial yield curve to have a non-bearing year.

2.2 The optimal replacement age for a two-age-class tree

Before studying the orchard problem, we need to know the optimal replacement age for a single tree, considered in isolation.

Assume a single perennial tree that lives for at most two periods, grown over an infinite horizon either on a 1-period or 2-period rotation. Let $\beta (< 1)$ be the discount factor. The net present value of the tree grown on a 1-period rotation is

$$\text{NPV}_1 = f_1 + \beta f_1 + \beta^2 f_1 + \beta^3 f_1 + \dots = \frac{f_1}{1 - \beta}$$

The net present value of the tree grown on a 2-period rotation is

$$\text{NPV}_2 = f_1 + \beta f_2 + \beta^2 f_1 + \beta^3 f_2 + \dots = \frac{f_1 + \beta f_2}{1 - \beta^2}$$

The difference between the two values is

$$\begin{aligned}
\text{NPV}_1 - \text{NPV}_2 &= \beta f_1 - \beta f_2 + \beta^3 f_1 - \beta^3 f_2 + \dots \\
&= \beta(f_1 - f_2) + \beta^3(f_1 - f_2) + \dots \\
&= \beta(f_1 - f_2)(1 + \beta^2 + \beta^4 + \dots) \\
&= \frac{\beta(f_1 - f_2)}{1 - \beta^2}
\end{aligned}$$

Hence if $f_1 > f_2$ the optimal replacement age is 1 (let N be the optimal replacement age, so $N = 1$). If $f_1 < f_2$ the optimal replacement age is 2 ($N = 2$). And if $f_1 = f_2$ ($N_1 = 1$; $N_2 = 2$).³

The optimal rotation age does not depend on discounting in this model. Discounting will affect the size of the net present value, but not the relative size of NPV_1 and NPV_2 . This is a special result, due to the assumption of a two-age-class model. In a more general model, the optimal single tree replacement age is a function of the discount factor, as shown by Mitra et al. (1991) in proposition 3.1.

2.3 The optimal replacement age for a three-age-class tree

Figure 2 shows the optimal replacement for a single three-age-class tree, normalizing the yield of the second age-class to one. The horizontal axis is the yield of the first age-class relative to the second, and the vertical axis is the yield of the third age-class relative to the second. The optimal replacement age also depends on the discount factor. The dotted line represents the boundary between the regions as the discount factor approaches zero, while the dashed line represents the boundary as the discount factor approaches one.

Regardless of the discount factor, there are parameter sets giving an optimal replacement age of three where the yield of the third age-class is less than the second age-class, that is with non-monotonic yield. Devadoss and Luckstead (2010) identified a declining final period as a key feature of perennials (recall quote from introduction). The analysis of Mitra et al. (1991) and Wan (1993)

³Theorem 3.1 in Mitra et al. (1991) states that there will either be one or two optimal cutting ages, N or N_1 and N_2 . The age(s) will occur during the declining section of the yield curve. If there are two optimal cutting ages, then they will be adjacent. That is $Q \leq N_1 \leq N_2 \leq N_1 + 1 \leq T$.

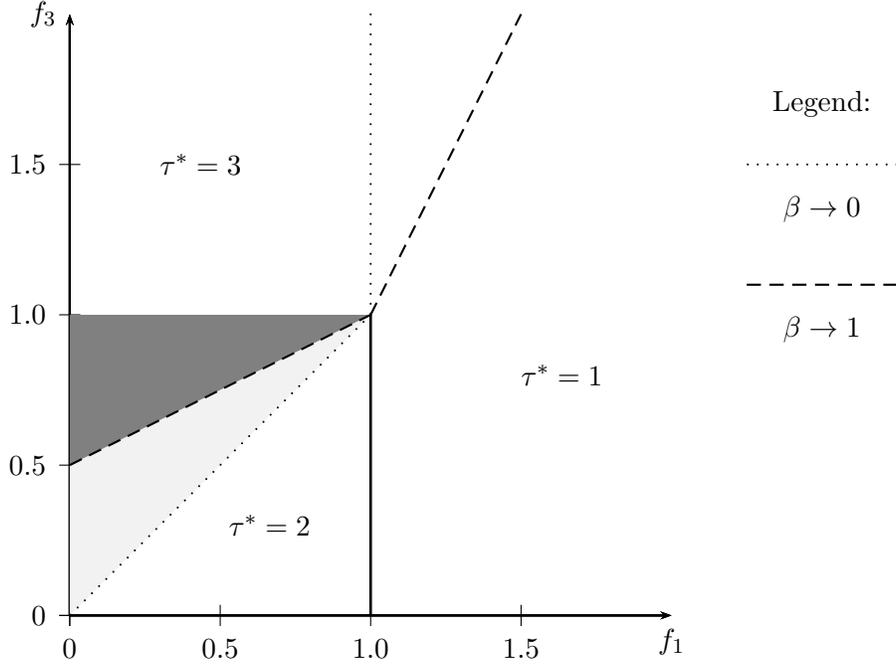


Figure 2: The optimal replacement age for a three-age-class tree. The yield of a medium tree has been normalized to 1, i.e. $f_2 = 1$. The location of the regions depend on the discount factor.

were restricted to the upper left quadrant of the figure, while our analysis of the three-age-class model encompasses the entire $i^* = 3$ region, i.e. the upper left quadrant and the dark gray region.

2.4 The optimal replacement age for an N -age-class tree

Proposition 3.1 from Mitra et al. (1991), adapted to use our notation, states:

Let the initial tree be of age i , where $0 \leq i \leq N$. There exist two integers i_1^* and i_2^* with $i_1^*, i_2^* \leq N$ and $R \leq i_1^* \leq i_2^* \leq i_1^* + 1$, such that the set of optimal cutting sequences is given precisely by those sequences which allow tree i , $i \geq 2$, to exist for i_1^* or i_2^* periods, and tree 1 to exist for $\max(0, i^* - i)$ periods, where $i^* = i_1^*$ or i_2^* . Moreover, i_1^* and i_2^* form the set of solutions to

$$\max_{0 \leq i^* \leq N} \frac{\sum_{i=1}^{i^*} f_i \beta^i}{1 - \beta^{i^*}}$$

3 A two-age-class orchard model

In this section we present a two-age-class, infinite horizon model of a perennial orchard. We find optimal stationary solutions for the model under three cases: young and old trees have identical yield, young trees have higher yield, and old trees have higher yield. In the case where old trees have higher yield, we find that the solution is generally cyclical, with one exception. We provide intuition for the existence of optimal cycles, and examine how the maximum cycle radius changes as function of the discount factor and the yields of young and old trees.

3.1 A two-age-class, infinite horizon orchard model

Salo and Tahvonen (2002) present a two-age-class, infinite horizon forestry model. We adapt the Salo and Tahvonen (2002) model to perennials by changing the production function from a point payoff to a flow payoff. Further, we adopt the aging constraint structure from Salo and Tahvonen (2004). That is, we do not include explicit choice variables for replanting—it is a reduced form dynamic optimization problem.

$$V(x_{10}, x_{20}) = \max_{\mathbf{x}_t} \sum_{t=0}^{\infty} \beta^t u(c_t) \tag{1}$$

subject to

$$c_t \equiv f_1 x_{1t} + f_2 x_{2t} \tag{2}$$

$$x_{2,t+1} \leq x_{1t} \tag{3}$$

$$x_{1t} + x_{2t} \leq L \quad (= x_{10} + x_{20}) \tag{4}$$

$$x_{st} \geq 0 \tag{5}$$

where x_{1t} is the quantity of land allocated to young trees in period t , x_{2t} is the quantity of land allocated to old trees in period t , f_1 is the productivity of young trees, f_2 is the productivity of

old trees,⁴ $u(c_t)$ is the benefit to the grower from growing/consuming fruit in period t . We assume that the benefit function exhibits diminishing marginal returns: $u'(\cdot) > 0$ and $u''(\cdot) < 0$. Finally, c_t is the total quantity of fruit harvested in period t .

This model is a reduced form version of the model of Salo and Tahvonen (2002) except that the definition of consumption, c_t , has been changed and an explicit total land constraint has been added. In their forestry model consumption in period t was the total mass of cut trees, i.e. the sum of young trees replanted plus old trees replanted. The consumption is calculated at the end of each tree's life, after the replanting decision has been made. In this model of perennial trees, consumption in period t is the sum of the fruit harvested from young and old trees, which does not depend on the number of trees replanted. Fruit is picked before replanting decisions are made and there is no direct benefit from a cut tree.

Forestry:

Inherit trees \rightarrow grow \rightarrow remove/replant \rightarrow consume \rightarrow bequeath trees to next period

Perennials:

Inherit trees \rightarrow grow \rightarrow consume \rightarrow remove/replant \rightarrow bequeath trees to next period

Equation 2 is the aging constraint.⁵ It constrains the number of old trees in the next period from exceeding the number of young trees in the current period. Old trees cannot be bought and planted, they must be grown from young trees.

This is a convex optimization problem since the objective function is strictly concave and the constraints are linear. Therefore any solution to the Karush-Kuhn-Tucker conditions will also be a solution to the constrained optimization problem posed above.

⁴These productivities are net of planting costs.

⁵Wan (1993) calls this constraint the *cross-vintage bound*. Other authors such as Salo and Tahvonen follow this terminology. We feel that *aging constraint* is more intuitive.

The corresponding Lagrangian function is

$$\mathcal{L} = \sum_{t=0}^{\infty} \beta^t [u(c_t) + \lambda_t(x_{1t} - x_{2,t+1}) + \psi_t(L - x_{1t} - x_{2t})]$$

The associated KKT conditions for $t \geq 1$ are

$$\beta^{-t} \frac{\partial \mathcal{L}}{\partial x_{1t}} = u'(c_t)f_1 + \lambda_t - \psi_t \leq 0 \tag{5}$$

$$\beta^{-t} \frac{\partial \mathcal{L}}{\partial x_{2t}} = u'(c_t)f_2 - \frac{\lambda_{t-1}}{\beta} - \psi_t \leq 0 \tag{6}$$

$$\lambda_t \geq 0; \lambda_t(x_{1t} - x_{2,t+1}) = 0 \tag{7}$$

$$\psi_t \geq 0; \psi_t(L - x_{1t} - x_{2t}) = 0 \tag{8}$$

$$x_{it} \geq 0; x_{it} \frac{\partial \mathcal{L}}{\partial x_{it}} = 0 \tag{9}$$

where λ_t is the Lagrangian multiplier corresponding to the cross-vintage bound constraint and ψ_t is the Lagrangian multiplier corresponding to the total land constraint. Because this is a reduced form specification of the problem, only the state variable—the area of old trees each period, x_t —must be chosen each period. There are no separate control variables.

There are two choice variables, and four constraints (c_t is considered a definition, not a constraint). Therefore, each period, there are two first order conditions (FOCs) and four complementary slackness conditions.

3.1.1 Interpreting ψ_t

The variable ψ_t is associated with the total land constraint. This model assumes that there is a fixed quantity of land available to the grower at the beginning of the problem and that area of land cannot be increased. The value of ψ_t represents the marginal increase in the value of the orchard in period t from a permanent marginal addition to total land.

If ψ_t is positive, the total land constraint is binding. An implication of this is that old trees are immediately replanted at the end of their lifespan. Land is not left fallow for any period of time. In this paper, we will proceed as if ψ_t is always positive. This will certainly be true if the

productivity coefficients of young and old trees are non-negative because the marginal utility of harvest is always positive. If the productivity of young trees is negative, which it might be if the productivity is interpreted as net of planting costs and planting costs are substantial, it is possible that additional land will not be valuable. However, if planting costs are this large, the grower has no incentive to grow the orchard at all and it will be abandoned as soon as all old trees have died.

3.1.2 Interpreting λ_t

The variable λ_t is associated with the cross-vintage bound, which constrains the area of land allocated to old trees in period $t + 1$ to not exceed the area of land allocated to young trees in period t . The value of λ_t represents the marginal increase in the value of the orchard in period t from a relaxation of the aging constraint between period t and $t + 1$. The constraint is relaxed by planting additional young trees in period t .

Equation 5 shows the marginal benefit in period t from increasing the land allocated to young trees. It consists of three components: the immediate marginal benefit from additional young trees in period t , the future marginal benefit from additional young trees in period t , and the immediate cost of using scarce land.

$$\beta^{-t} \frac{\partial \mathcal{L}}{\partial x_{1t}} = \underbrace{u'(c_t) f_1}_{\text{Current marginal benefit}} + \underbrace{\lambda_t}_{\text{Future marginal benefit}} - \underbrace{\psi_t}_{\text{Marginal cost of land}}$$

Growing an additional young tree in period t allows additional old trees to be grown in period $t + 1$. Assuming an interior solution ($x_{1t}, x_{2t} > 0$) and rearranging equation 6 we see that the net marginal benefit of an additional old tree in period $t + 1$ (valued in period t) is the marginal utility from an old tree, less the cost of allocating the land to that tree

$$\lambda_t = \beta(u'(c_{t+1}) f_2 - \psi_{t+1})$$

When the total land constraint binds ($\psi_t > 0$), the aging multiplier becomes

$$\lambda_t = \beta(u'(c_{t+1})f_2 - u'(c_{t+1})f_1 - \lambda_{t+1})$$

that is, the marginal utility from an additional old tree in period $t + 1$ is the marginal benefit of that old tree, less the marginal benefit of a young tree in period $t + 1$. Because the total land constraint binds, an additional old tree in period $t + 1$ must be grown at the expense of a young tree in period $t + 1$. We will use this form of λ_t when analyzing the stationary solutions to the two-age-class model.

3.2 Stationary solutions to the model

We will identify stationary solutions to the two-age-class, infinite horizon orchard model. A stationary solution is one where either the land allocation is constant over time, or the land allocation is cyclical, returning to the same state after finite time. We will begin by identifying a necessary and sufficient condition for a two-period cyclical solution to the model. The steady-state (one period cycle) will emerge as a special case of this condition. We restrict our attention to interior solutions with a binding total land constraint, i.e. $x_{2t} > 0$ and $L - x_{1t} - x_{2t} = 0$, which implies that the Euler inequality (equation 6) is satisfied with equality and that ψ_t is strictly positive for all t .

In a two-period cycle, the land allocation will repeat every two periods, so $x_{2t} = x_{2,t+2}$. If the land allocations repeat, the harvest values must repeat too, $c_t = c_{t+2}$. To show that such a cycle is optimal, we must find a set of non-negative λ_t to satisfy the KKT conditions.

Throughout this analysis we assume that old trees are more productive than young trees,⁶ $f_2 > f_1$, and that $0 < \beta < 1$.

Proposition 1. *For all $f_1 < f_2$ and $0 < \beta < 1$, there exist cyclical solutions to the two-age-class orchard management problem only if $\beta \leq \frac{u'(c_{t+1})}{u'(c_t)} \leq \frac{1}{\beta}$*

Proof. See page 38 □

⁶If $f_1 > f_2$ there cannot be any old trees in a stationary solution to the problem. This yield assumption implies that equation 6 must be strictly negative, which implies that x_{2t} must be zero by the complementary slackness condition 9.

The maximum radius of the optimal cycles is discussed in section 3.4.

Corollary 2. *For all $f_1 < f_2$ and $0 < \beta < 1$, the balanced orchard is a solution to the two-age-class orchard management problem (i.e. $x_{2t} = \frac{L}{2}$)*

Proof. See page 39 □

3.3 Intuition for the existence of cycles

To understand whether a cycle is stable or not, consider the marginal value of a marginal deviation from the cycle. Assume that the orchard at time t is being managed with the following cyclical land allocation, $\{\mathbf{x}_t, \mathbf{x}_{t+1}\} = \{(a, L - a), (L - a, a), \dots\}$. The associated consumption sequence is $\{c_t, c_{t+1}, \dots\} = \{f_1 a + f_2(L - a), f_1(L - a) + f_2 a, \dots\}$. From the perspective of period t , the value of this orchard is

$$\beta^{-t} V_t = u(c_t) + \beta u(c_{t+1}) + \beta^2 u(c_t) + \dots$$

This cyclical allocation will be optimal if the grower has no incentive to adjust the allocation. There are two ways of adjusting the allocation: the grower can increase the number of young trees in period $t + 1$ by replacing young trees at the end of period t , or the grower can increase the number of young trees in period $t + 2$ by replacing young trees at the end of period $t + 1$. The grower takes the allocation in period t as given.

If the marginal change in value from a marginal increase in young trees in both period $t + 1$ and $t + 2$ is non-positive, then the cycle will be optimal.

Focusing on the change in young trees in period t , the marginal change in value is

$$\begin{aligned} \beta^{-t} \frac{\partial V_t}{\partial x_{1,t+1}} &= 0 + \beta u'(c_{t+1})(f_1 - f_2) + \beta^2 u'(c_{t+2})(f_2 - f_1) + \dots & (10) \\ &= \frac{\beta(f_2 - f_1)}{1 - \beta^2} (\beta u'(c_{t+2}) - u'(c_{t+1})) \\ &= \frac{\beta(f_2 - f_1)}{1 - \beta^2} (\beta u'(c_t) - u'(c_{t+1})) & \text{(because } c_t = c_{t+2}) \end{aligned}$$

There is no incentive to adjust the area of young trees in period $t+1$ if this expression is non-positive

$$\beta^{-t} \frac{\partial V_t}{\partial x_{1,t+1}} \leq 0 \Leftrightarrow \frac{u'(c_{t+1})}{u'(c_t)} \geq \beta$$

Similarly for increasing young trees in period $t+2$

$$\beta^{-t} \frac{\partial V_t}{\partial x_{1,t+2}} \leq 0 \Leftrightarrow \frac{u'(c_{t+1})}{u'(c_t)} \leq \frac{1}{\beta}$$

Combining these inequalities gives the same restriction on the ratio of marginal utilities derived from the KKT conditions in proposition 1.

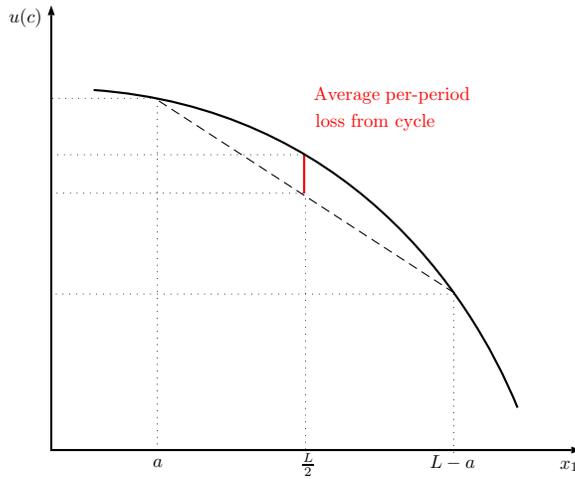


Figure 3: Cyclical production leads to a loss in average utility.

Figure 3 shows the utility as a function of the area allocated to young trees, x_1 , assuming all land is used. This figure also assumes that $a < \frac{L}{2}$. When there are few young trees, $x_1 = a$, there are many old trees, $x_2 = L - a$, total utility is high, and marginal utility is low; *vice versa* when there are many young trees. Proposition 1 requires the ratio of the marginal utilities each period to be sufficiently close to 1 and the land allocations sufficiently close to $\frac{L}{2}$.

More broadly, the grower faces competing incentives generated by time preference and the preference for consumption smoothing. A grower with a positive discount rate is willing to forgo a larger quantity of future consumption to for a smaller increase in present consumption. A grower with a preference for consumption smoothing receives greater total utility from a consumption

stream with less year-to-year variation.

In the orchard model, these two preferences can be in tension or work in the same direction. The grower can increase utility in period $t + 2$ by replacing young trees at the end of period t , thereby increasing the area of old trees and thus consumption in period $t + 2$. This action, however, reduces utility in period $t + 1$ because there are now fewer old trees that period. The discounting effect determines whether the utility gain in period $t + 2$ is large enough to compensate for the utility loss in period $t + 1$. This operation may increase or decrease the year-to-year variation in consumption, depending on the age-structure of the orchard in period t . If the replacement of young trees leads to less variation, then the utility gain from less variation is weighed against the losses from discounting. If, on the other hand, the replacement of young trees leads to more variation, there is no tension between the incentives; replacing the young trees early would lead to an immediate loss for no future gain. We will see how the initial conditions affect these trade-offs in section 3.5.

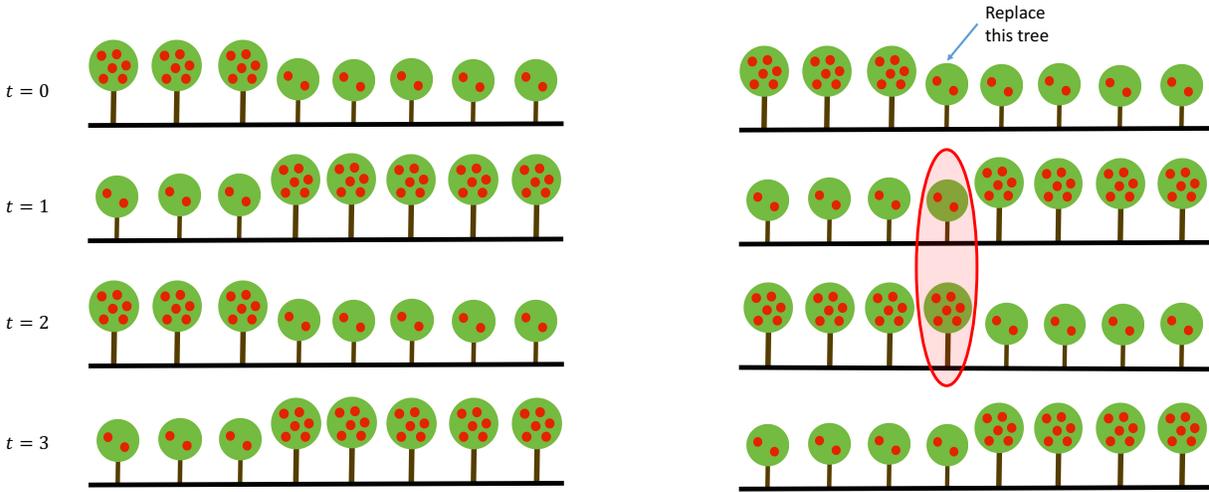
Figure 4 illustrates an example where the grower has eight trees each year with $x_{1,0} = \frac{5}{8}$. This grower is facing discrete replacement decisions, but it serves to illustrate the intuition behind whether to replace trees early, which is the same as the marginal case above. Subfigure 4a shows the trajectory of this orchard if the grower follows Faustmann replacement only and does not engage in smoothing. In contrast, subfigure 4b shows the same initial orchard where the grower engages in smoothing by replacing one of the young trees. By engaging in early replacement, the grower achieves a balanced orchard in period 2, but has lower production in period 1. Whether this operation increases the grower's utility depends on the discount factor.

Figure 5 shows the utility and net utility to the grower from these two orchards when the discount rate is zero. With zero discount rate, the grower only has an incentive to smooth consumption by maximizing the average harvest. The utility level of the balanced orchard is higher than the average utility from the unbalanced orchard in period one and two. This is due to the curvature of the utility function and Jensen's inequality. Hence the utility loss in period one (-0.3) from early replanting is outweighed by the utility gain in period two from the smoother harvest (0.45).

As the discount rate increases more weight is placed on the loss in period one and less is placed

on the gain in period two. Figure 6 shows the utility and net utility to the grower when the discount factor is 0.5. With this discount factor, the present value of the utility loss in period one is -0.015 while the gain in period two is only 0.011. The loss outweighs the gain and the grower would not replace the young tree early, keeping the unbalanced orchard in figure 4a for the remainder of the time horizon.

Unlike this example, the land allocation in the orchard model is a continuous variable. If, in the initial condition, replacing the marginal young tree is worthwhile, the farmer can continue replacing young trees until the marginal benefit from replacing the young tree early equals the marginal benefit from leaving it to grow. This is the idea behind equation 10.



(a) Without smoothing – no short term loss, but long run loss from larger cycle.

(b) With smoothing – grower forgoes an old tree in period 2, but smoother in the long run.

Figure 4: The age-structure of an orchard without and with smoothing.

3.4 Comparative statics of cycle radius

The inequalities in proposition 1 define a region within which it is optimal for the grower to maintain a cyclical land allocation. How does the size of this region change with the discount factor, the total area of land, and the relative productivity of young and old trees. First we need to identify this region, and then calculate comparative statics to examine how it changes with parameters.

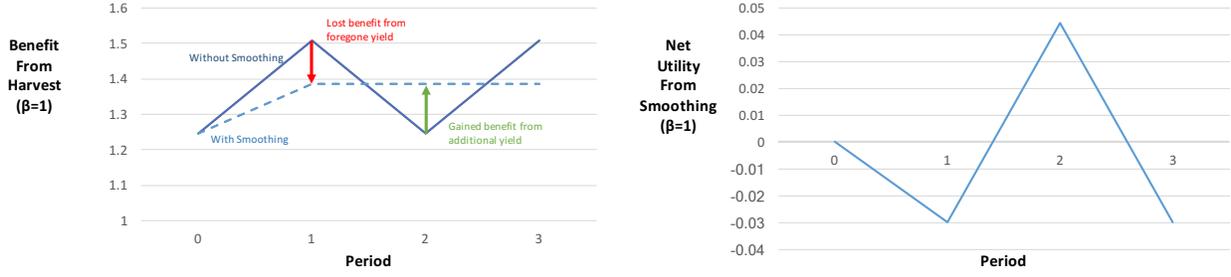


Figure 5: With no discounting, the gain in period 2 outweighs the loss in period 1.

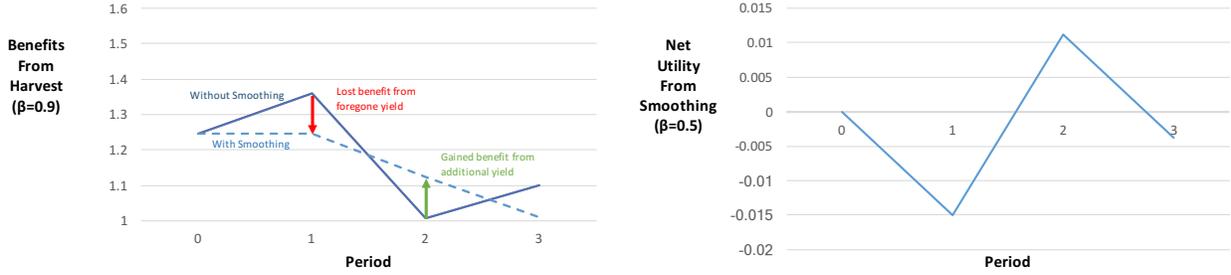


Figure 6: With a sufficiently high discount rate, the loss in period 1 outweighs the gain in period 2.

Let $c(x_{1t})$ be the harvest when x_{1t} units of land are planted with young trees. Using this definition, the inequalities from proposition 1 become

$$\beta \leq \frac{u'(c(x_{1,t+1}))}{u'(c(x_{1t}))} \leq \frac{1}{\beta}$$

Without loss of generality, we assume that the cycle peak occurs in period t , so $u'(c(x_{1,t+1})) > u'(c(x_{1t}))$. Hence the right hand inequality is the relevant inequality. Requiring this inequality to be satisfied with equality gives the largest difference in marginal utilities such that a cycle will be optimal. Writing the allocation of young trees in terms of deviations from the balanced orchard, $x_{1t} = \frac{L}{2} - \phi$ and $x_{1,t+1} = \frac{L}{2} + \phi$, implicitly defines the maximum cycle radius given the parameters of the model.

$$\frac{u'(c(\frac{L}{2} - \phi))}{u'(c(\frac{L}{2} + \phi))} = \beta \tag{11}$$

Using this implicit definition of the maximum cycle radius, we can use the implicit function theorem

to find the comparative statics of the maximum cycle radius with respect to the parameters of the orchard management problem.

Proposition 3. *The comparative statics of the maximum cycle radius with respect to the parameters β , f_1 and f_2 are presented in table 1. Where $A(x) = \frac{-u''(x)}{u'(x)}$, the Arrow-Pratt measure of absolute risk aversion.*

Proof. See page 39 □

α	$\frac{\partial \phi}{\partial \alpha}$
β	(< 0)
f_1	$(> 0) \Leftrightarrow \frac{A(c(\frac{L}{2}+\phi))}{A(c(\frac{L}{2}-\phi))} < \frac{(\frac{L}{2}+\phi)}{(\frac{L}{2}-\phi)}$
f_2	$(< 0) \Leftrightarrow \frac{A(c(\frac{L}{2}+\phi))}{A(c(\frac{L}{2}-\phi))} < \frac{(\frac{L}{2}-\phi)}{(\frac{L}{2}+\phi)}$
L	$(> 0) \Leftrightarrow \frac{A(c(\frac{L}{2}+\phi))}{A(c(\frac{L}{2}-\phi))} > 1$

Table 1: Signs of comparative statics of cycle radius, ϕ , with respect to β , f_1 , and f_2 .

The level of absolute risk aversion in the utility function affects the comparative statics of the maximum cycle radius. For example, for a function with constant absolute risk aversion, maximum cycle radius would always increase with f_1 and decrease with f_2 .

3.5 Transitioning to the cycle

Having identified the long-run steady state of the orchard, the question arises: what happens in the short run? Does the orchard transition quickly to the steady-state, or only asymptotically? Proposition 4 defines the optimal transition rule between periods, which identifies the trajectory to the steady-state from any initial orchard.

Proposition 4. *Assuming the land constraint binds every period and letting x_t be the land allocated*

to old trees in period t , the optimal transition rule is given by

$$x_{t+1} = P(x_t) = \begin{cases} \frac{L}{2} + \phi & \text{for } x_t \in [0, \frac{L}{2} - \phi) \\ L - x_t & \text{for } x_t \in [\frac{L}{2} - \phi, \frac{L}{2} + \phi] \\ L - x_t & \text{for } x_t \in (\frac{L}{2} + \phi, L] \end{cases}$$

where ϕ is the maximum cycle radius, as defined in equation 11. See page ?? for proof.

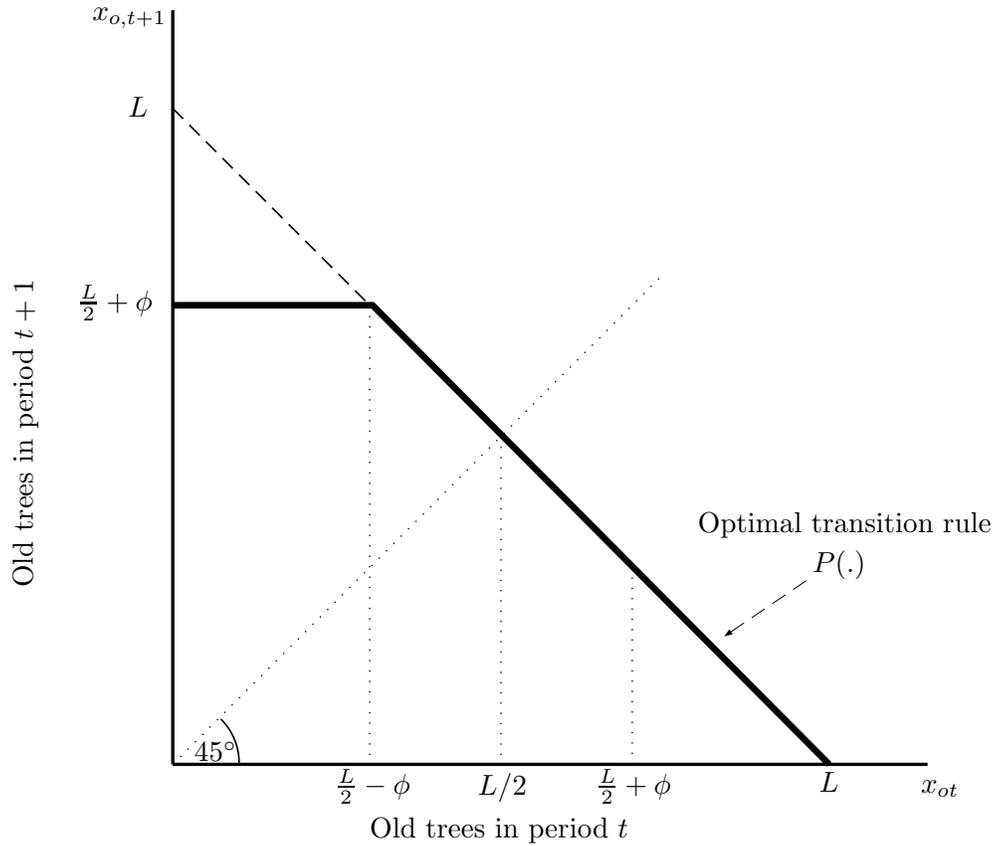


Figure 7: Transition map showing the (conjectured) optimal transition rule for a two age class orchard.

Figure 7 shows the optimal transition map between old trees in period t and old trees in period $t + 1$. The horizontal axis denotes the area allocated to old trees in period t and the vertical axis denotes old trees in period $t + 1$. This diagram is drawn assuming that the total land constraint binds every period. The downward sloping dashed line represents the aging constraint, $x_{2,t+1} \leq L - x_{2t}$.

The bold black line represents the optimal transition rule, showing the optimal area allocated to old trees in period $t + 1$ given an allocation of old trees in period t .

For allocations of old trees larger than $L - \phi$, the aging constraint binds. However, when there are few old trees in period t (i.e. $x_{2t} < L - \phi$), some young trees are replaced at the end of period t so $x_{2,t+1} < L - x_{2t}$. When there are many young trees, the opportunity cost of replacing a young tree early is low.

An implication of this diagram is that the two-age-class orchard will converge to an optimal cycle in at most two periods. If the initial orchard has $x_{2t} < L - \phi$, then an optimal cycle will be reached in the next period. If the initial orchard has $x_{2t} > L + \phi$, then an optimal cycle will be reached in two periods.

For an orchard, the optimal transition rule is horizontal for low allocations of old trees in period t , or downward sloping along the aging constraint for higher allocations of old trees. Contrast this to the optimal transition rule in a two-age-class forest, which, for low allocations of old trees, is upward sloping before it intersects with the aging constraint (Salo and Tahvonen, 2002, fig. 1).

This difference arises because of the different timing of benefits in orchard and forestry models. In forestry, benefits are obtained after trees are cut. If there is no cutting, there are no benefits. In Salo and Tahvonen's model, all old trees are automatically harvested at the end of the period. If there are very few old trees, there will be very little automatic harvesting, so the marginal utility of harvesting is very high. This induces the additional harvesting of young trees to increase the total harvest in period t , outweighing the future benefits of leaving the timber to grow for another year. Thus there are two incentives to cut young trees: increasing current consumption, and smoothing future timber harvests.

On the other hand, in the orchard model there is no direct benefit to replacing a tree; the benefit from the fruit is obtained before the tree is replaced. Therefore the only incentive to replace a young tree early comes from the benefits of a smoother harvest trajectory in the future. There is no benefit in replacing more young trees than is necessary to enter the optimal cycle region.

The two-age-class forestry model converges to an optimal cycle in finite time because of the upwards sloping portion of the optimal transition rule, but the time to convergence depends on

the model's parameters. In contrast, the orchard model converges to the optimal cycle region in at most two periods, regardless of the parameters (as long as $f_1 < f_2$).

4 A three-age-class orchard model

We now extend the model to include a third age-class. In this model, trees live for three periods before dying. The three periods are labeled: young, mature, and old trees. As before, the grower's objective is to maximize the discounted benefits from the stream of harvests from each type of tree over an infinite time horizon.

The statement of the problem is:

$$V(x_{10}, x_{20}, x_{30}) = \max_{\mathbf{x}_t} \sum_{t=0}^{\infty} \beta^t u(c_t) \quad (12)$$

subject to

$$c_t \equiv f_1 x_{1t} + f_2 x_{2t} + f_3 x_{3t} \quad (13)$$

$$x_{2,t+1} \leq x_{1t} \quad (13)$$

$$x_{3,t+1} \leq x_{2t} \quad (14)$$

$$x_{1t} + x_{2t} + x_{3t} \leq L \quad (15)$$

$$x_{it} \geq 0 \quad (16)$$

The Lagrangian of the reduced problem is

$$\mathcal{L} = \sum_{t=0}^{\infty} \beta^t [u(c_t) + \lambda_{1t}(x_{1t} - x_{2,t+1}) + \lambda_{2t}(x_{2t} - x_{3,t+1}) + \psi_t(L - x_{1t} - x_{2t} - x_{3t})]$$

This leads to the KKT conditions for $t \geq 1$

$$\beta^{-t} \frac{\partial \mathcal{L}}{\partial x_{1t}} = u'(c_t) f_1 + \lambda_{1t} - \psi_t \leq 0 \quad (17)$$

$$\beta^{-t} \frac{\partial \mathcal{L}}{\partial x_{2t}} = u'(c_t) f_2 - \frac{\lambda_{1,t-1}}{\beta} + \lambda_{2t} - \psi_t \leq 0 \quad (18)$$

$$\beta^{-t} \frac{\partial \mathcal{L}}{\partial x_{3t}} = u'(c_t) f_3 - \frac{\lambda_{2,t-1}}{\beta} - \psi_t \leq 0 \quad (19)$$

$$\lambda_{st} \geq 0; \lambda_{st}(x_{st} - x_{s+1,t+1}) = 0 \quad \text{for } s \in 1, 2 \quad (20)$$

$$\psi_t \geq 0; \psi_t(L - x_{1t} - x_{2t} - x_{3t}) = 0 \quad (21)$$

$$x_{it} \geq 0; x_{it} \frac{\partial \mathcal{L}}{\partial x_{it}} = 0 \quad \text{for } i \in 1, 2 \quad (22)$$

4.1 Stationary solutions to the 3AC model

The focus of the following analysis will be on the existence and comparative statics of cyclical stationary optimal trajectories. Three periods is the longest possible cycle, since there are only three age-classes. In a three period cycle, the cross-vintage bounds hold with equality, and the land allocation in period $t + 3$ is equal to the land allocation in period t .

Assuming the derivatives with respect to the state variables hold with equality (interior optimum), we can use equations 18 and 19 to derive expressions for the Lagrangian multipliers for each period in the cycle.

$$\begin{bmatrix} \lambda_{1t} \\ \lambda_{1,t+1} \\ \lambda_{1,t+2} \end{bmatrix} = \frac{\beta}{1 - \beta^3} \begin{bmatrix} \beta^2(f_1 - f_3) & (f_2 - f_1) & \beta(f_3 - f_2) \\ \beta(f_3 - f_2) & \beta^2(f_1 - f_3) & (f_2 - f_1) \\ (f_2 - f_1) & \beta(f_3 - f_2) & \beta^2(f_1 - f_3) \end{bmatrix} \begin{bmatrix} u'(c_t) \\ u'(c_{t+1}) \\ u'(c_{t+2}) \end{bmatrix} \quad (23)$$

$$\begin{bmatrix} \lambda_{2t} \\ \lambda_{2,t+1} \\ \lambda_{2,t+2} \end{bmatrix} = \frac{\beta}{1 - \beta^3} \begin{bmatrix} \beta^2(f_2 - f_3) & (f_3 - f_1) & \beta(f_1 - f_2) \\ \beta(f_1 - f_2) & \beta^2(f_2 - f_3) & (f_3 - f_1) \\ (f_3 - f_1) & \beta(f_1 - f_2) & \beta^2(f_2 - f_3) \end{bmatrix} \begin{bmatrix} u'(c_t) \\ u'(c_{t+1}) \\ u'(c_{t+2}) \end{bmatrix} \quad (24)$$

A sufficient condition for a three period interior cycle to be optimal is that all six Lagrange multipliers must be non-negative. A sufficient condition for this is that the Faustmann age for the related single tree replacement problem must be 3.

For a perennial crop, the Faustmann age, i^* , is defined by

$$\frac{\sum_{s=1}^{i^*} \beta^{s-1} f_s}{1 - \beta^{i^*}} \geq \frac{\sum_{s=1}^i \beta^{s-1} f_s}{1 - \beta^i} \quad \forall i \in \{1, \dots, N\}$$

The Faustmann age is unique if this condition is satisfied with strict inequality.

For a three-age-class model, a unique Faustmann age of three implies two inequalities

$$\begin{aligned} \frac{f_1}{1 - \beta} < \frac{f_1 + \beta f_2 + \beta^2 f_3}{1 - \beta^3} &\Leftrightarrow f_1 < \frac{f_2 + \beta f_3}{1 + \beta} \\ \frac{f_1 + \beta f_2}{1 - \beta^2} < \frac{f_1 + \beta f_2 + \beta^2 f_3}{1 - \beta^3} &\Leftrightarrow f_1 < f_3(1 + \beta) - \beta f_2 \end{aligned}$$

Assuming $i^* = 3$, in a balanced (non-cyclical) stationary orchard, one third of the land is allocated to each of the three age classes each period. A constant land allocation leads to constant utility and to constant marginal utility. Let $\bar{c} = \frac{f_1 + f_2 + f_3}{3}$, the production of the balanced orchard. The λ 's are therefore constant, reducing the sufficiency conditions to

$$\begin{aligned} \lambda_1 = \frac{\beta u'(\bar{c})}{1 - \beta^3} ((f_2 - f_1) + \beta(f_3 - f_2) + \beta^2(f_1 - f_3)) \geq 0 &\Leftrightarrow f_1 \leq \frac{f_2 + \beta f_3}{1 + \beta} \\ \lambda_2 = \frac{\beta u'(\bar{c})}{1 - \beta^3} ((f_3 - f_1) + \beta(f_1 - f_2) + \beta^2(f_2 - f_3)) \geq 0 &\Leftrightarrow f_1 \leq f_3(1 + \beta) - \beta f_2 \end{aligned}$$

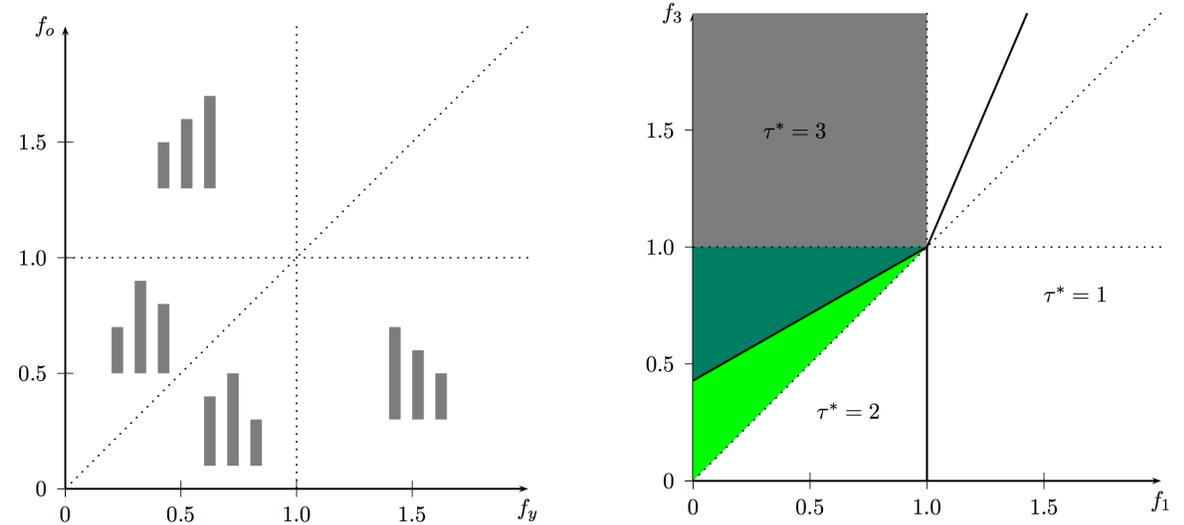
Therefore we can see how assuming a unique Faustmann age of three guarantees the optimality of the balanced stationary orchard

$$\begin{aligned} f_1 < \frac{f_2 + \beta f_3}{1 + \beta} &\Rightarrow \lambda_1 > 0 \\ f_1 < f_3(1 + \beta) - \beta f_2 &\Rightarrow \lambda_2 > 0 \end{aligned}$$

Since both multipliers are strictly positive at the balanced stationary orchard, by the continuity of 23 and 24 there exists some set of deviations ϕ_1 , ϕ_2 and ϕ_3 , where $\phi_1 + \phi_2 + \phi_3 = 0$ and at least

one ϕ_s is non-zero, such that the cycle defined by $\mathcal{O} = \{\frac{1}{3} + \phi_1, \frac{1}{3} + \phi_2, \frac{1}{3} + \phi_3\}$ is also optimal.

Therefore when there are three age-classes and for all $\beta \in (0, 1)$ there exist cyclical trajectories of period 3 that are optimal. Denote the set of allocations from which the optimal trajectory is a cycle as K_0 .



(a) The space of three age-class yield curves, with f_2 normalized to 1.

(b) The Faustmann age as a function of the yield curve. The gray square represents the yield curves studied by Mitra et al. (1991). The light green ($i^* = 2$) and dark green ($\tau^* = 3$) regions represent the non-monotonic yield curves that can be analyzed by our model.

4.2 Uniform convergence to the cycle region

Theorem 5. *For all $\mathbf{x}_0 \notin K_0$, the optimal trajectory converges uniformly to K_0 .*

4.3 Example trajectories: Numerical comparative statics for three-age-class model

This section uses a running horizon algorithm to approximate the optimal trajectory for a three-age-class model (see appendix A for more details on the algorithm. We use these approximations to develop numerical comparative statics with respect to the yield of old trees and the discount factor.

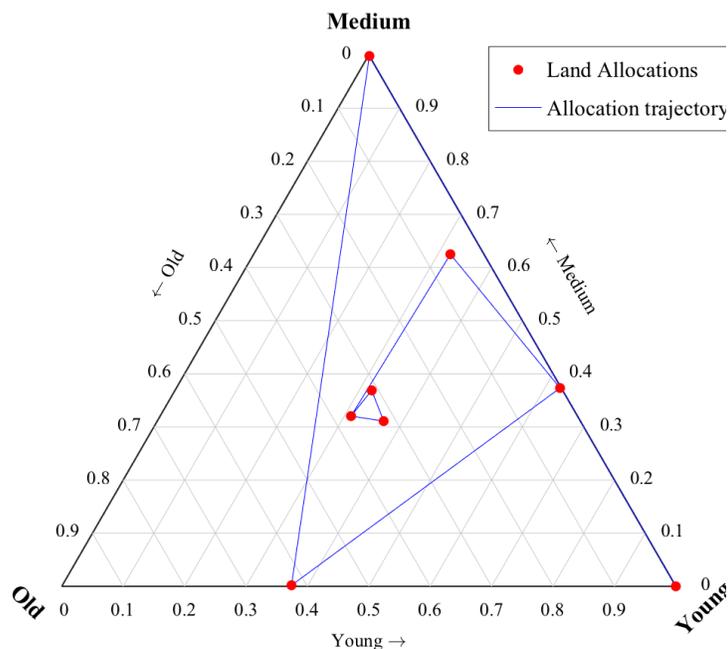


Figure 9: Example trajectory starting from all young trees with $\beta = 0.75$, $f = (1, 3, 2)$

Figure 9 shows an example trajectory beginning with all young trees (right corner of triangle) assuming $\beta = 0.75$, $f = (1, 3, 2)$. The red dots show the land allocation in each period, and the blue lines connecting them show the transitions.

In the initial period there are all young trees. In period one, all the young trees have matured to become medium trees. In period two, just under two thirds of the medium trees become old trees, and just over one third are replaced before their Faustmann age of three. In period three, there are only young and medium trees. No trees are replaced early. In period four, all young trees become medium trees, but most medium trees are cut (before their Faustmann age), leaving only a few old trees. In period five, the trajectory enters the optimal cycle, centered on the balanced orchard.

Figure 10 shows numerical comparative statics for trajectories in the three-age-class model. Like the two-age-class model, the cycle region shrinks in size as the discount factor increases. The cycle radius is largest when the the yield of medium and old trees is equal.

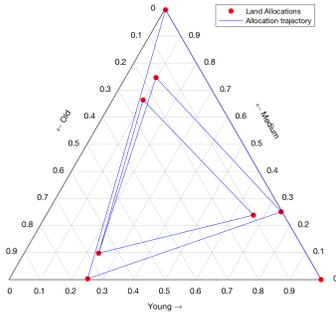
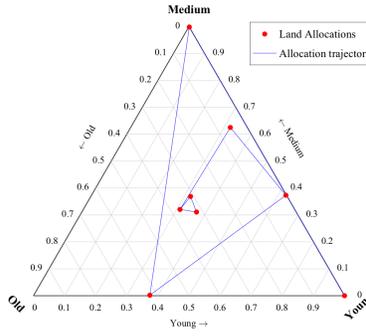
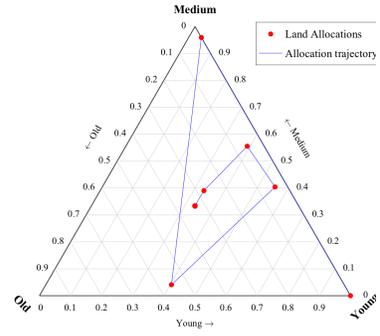
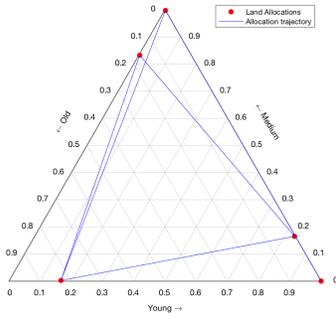
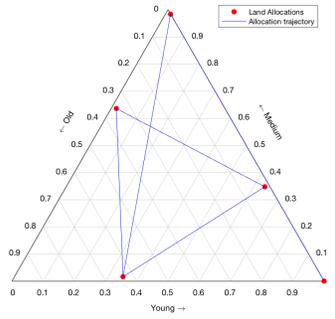
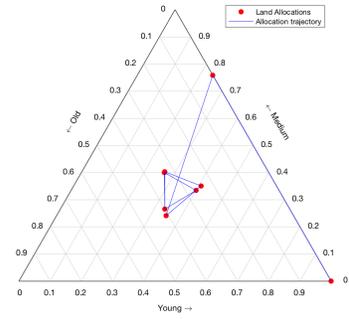
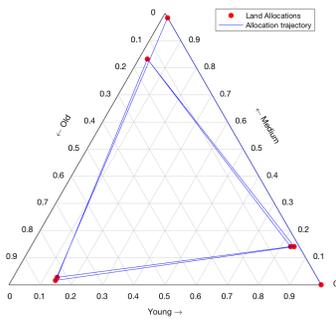
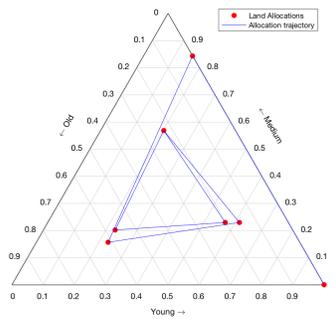
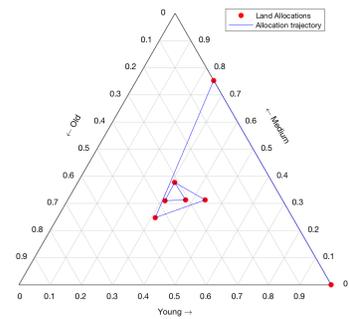
(a) $\beta = 0.5, f = [1 \ 3 \ 2]$ (b) $\beta = 0.75, f = [1 \ 3 \ 2]$ (c) $\beta = 0.95, f = [1 \ 3 \ 2]$ (d) $\beta = 0.5, f = [1 \ 3 \ 3]$ (e) $\beta = 0.75, f = [1 \ 3 \ 3]$ (f) $\beta = 0.95, f = [1 \ 3 \ 3]$ (g) $\beta = 0.5, f = [1 \ 3 \ 6]$ (h) $\beta = 0.75, f = [1 \ 3 \ 6]$ (i) $\beta = 0.95, f = [1 \ 3 \ 6]$ 

Figure 10: Three-age-class optimal orchard trajectories. Columns hold discount rate constant. Rows hold the yield curve constant. Each figure starts at with an initial orchard composed entirely of young trees, $\mathbf{x}_0 = (1 \ 0 \ 0)$

4.4 Numerical comparative statics of orchard value

Figure ?? shows numerical comparative statics of orchard value. Each subfigure represents a value surface, showing the value of the optimal program for each initial condition. The red dot shows the highest value orchard for each set of parameters. For low discount factors, the optimal orchard is on the boundary of the simplex. As the discount factor increases, the optimal orchard converges towards the balanced orchard. This convergence appears to occur more quickly when the yield of old trees is greater than the yield of medium trees, that is when the yield curve is monotonically increasing. Additionally, increasing the yield of old trees increases the allocation of land to old trees in the optimal orchard.

5 Discussion

5.1 Production cycles

A notable feature of perennial crops is that they exhibit boom and bust cycles in their production, increasing the risk to farmers of investing in these crops.

It has been conjectured that better price forecasts could reduce or eliminate acreage cycles in perennial crops. For example, in the case of lemons in California, French and Bressler (1962, p. 1036) stated: “If ‘better’ knowledge, including the realization that there is a cycle, leads to more realistic forecasts, then the cycle would be moderated or eliminated.” Knapp (1987) has advanced an alternative hypothesis: acreage cycles are an inherent feature of perennial crop production systems, arising from the biology of the crop causing lags in production, rather than from imperfect information and foresight on the part of the growers.

Our work shows that production cycles can be a feature of optimally managed perennial crop production in a setting with perfect foresight (no uncertainty). This suggests that improved crop forecasts would not eliminate perennial crop production cycles.

6 Conclusion

In this paper we extended the existing work on the optimal management of age-structured orchards in two key ways. First, we extended the results for a grower with concave utility and trees with a monotonically increasing yield by identifying the optimal transition trajectory from an arbitrary initial orchard to the long-run, cyclical, steady-state. Furthermore, we characterized the cycle regions and showed how its radius changes as a function of the economic and technological parameters of the grower's problem. Second, we extended the theoretical results on the long-run steady-state of an optimally managed orchard to a setting with a non-monotonic yield curve, grounding the results in a more realistic tree technology.

In both settings, the grower faces a trade-off between discounting and income smoothing. The benefits of smoother production approach zero as the orchard approaches the balanced orchard, but the losses due to discounting remain strictly positive, so the grower always accepts a level of cyclical production (unless the grower began with a balanced orchard).

These results answer a natural conjecture about the optimality of the balanced orchard. Even acknowledging that the optimal trajectory starting from an initial condition away from the balanced orchard does not lead to the balanced orchard, one might conjecture that the balanced orchard is the optimal initial condition. That is, if the grower had free choice of initial orchard, knowing that the orchard was to be managed optimally thereafter, would the grower pick the balanced orchard? One might be tempted to answer yes, because the balanced orchard maximizes the per-period average utility of the grower. However, in the presence of discounting, our numerical results show that the grower prefers an initial orchard with a larger fraction of land allocated to higher yielding age-classes. Again, discounting is at work, leading the grower to prefer a short-run increase in utility at the expense of a long-run increase in average utility. For a monotonically increasing yield curve, our numerical results suggest that the balanced orchard is increasingly preferred as the discount factor approaches one. This is consistent with the 'golden rule' of capital accumulation from undiscounted capital theory. However, for the non-monotonic yield case, it is not clear that the balanced orchard is preferred as the discount factor approaches one. More research is required to determine, in the

case of non-monotonic yields, whether the optimal initial orchard simply converges more slowly to the balanced orchard, or that the golden rule is not followed in this case.

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A Running Horizon Algorithm

The running horizon algorithm approximates the solution to an infinite horizon dynamic problem by calculating the solution to a sequence of finite horizon problems. This description is adapted from Franklin (2012) and Salo and Tahvonen (2004). The relevant parts of their papers are included in the appendix.

We wish to find a solution to the non-linear programming problem

$$V(\mathbf{x}_0) = \max_{\{\mathbf{x}_t\}_{t=1}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t) \quad \text{s.t. constraints}$$

Numerically solving this problem directly requires approximating the value function, a computationally intensive and unstable process.

Alternatively, we can repeatedly solve the finite horizon analog of the infinite horizon problem to obtain an approximation of the infinite horizon solution. The finite horizon analog is given by:

$$V^T(\mathbf{x}_0) = \max_{\{\mathbf{x}_t\}_{t=1}^T} \sum_{t=0}^T \beta^t u(c_t) \quad \text{s.t. constraints}$$

Solving this problem is relatively straightforward, and can be directly implemented with a non-linear numerical optimization algorithm, e.g. the `fmincon` command in MATLAB.

The optimal sequence in this finite horizon problem is $\{\mathbf{x}_t^*(T, \mathbf{x}_0)\}_{t=1}^T$. The parentheses following \mathbf{x}_t denote the dependence of the solution on the length of the finite horizon, T , and the initial condition \mathbf{x}_0 .

The running horizon algorithm generates a vector of length S , approximating the first S terms of the infinite horizon problem, $\{\hat{\mathbf{x}}_t\}_{t=1}^S \approx \{\mathbf{x}_t^{(\infty, \mathbf{x}_0)}\}_{t=1}^S$. The choice of S may affect the accuracy of the approximation, since errors in the approximation in the early terms will propagate to the later terms. For the orchard management problem, we want to pick an S large enough for the infinite horizon approximation to reach the steady state.

The first term of the approximation vector is the first term of the solution to the T period problem, starting from \mathbf{x}_0 . That is, $\hat{\mathbf{x}}_1 = \mathbf{x}_1^*(T, \mathbf{x}_0)$. The second term of the approximation vector

is generated by solving the T period problem, using $\hat{\mathbf{x}}_1$ as the initial condition, and taking the first term of the solution to this finite horizon problem. That is, $\hat{\mathbf{x}}_2 = \mathbf{x}_1^*(T, \hat{\mathbf{x}}_1)$. The remaining $S - 2$ terms of the approximation vector are generated in a similar manner. That is, $\hat{\mathbf{x}}_s = \mathbf{x}_1^*(T, \hat{\mathbf{x}}_{s-1})$.

This algorithm works because the first period solution to the finite horizon problem approaches the first period solution to the infinite horizon problem as the finite horizon approaches infinity.

$$\lim_{T \rightarrow \infty} \mathbf{x}_1^{(T, \mathbf{x}_0)} = \mathbf{x}_1^{(\infty, \mathbf{x}_0)}$$

This occurs because, with positive discounting, the behavior of the finite horizon problem at the terminal time has a diminishing effect on the first period choice as the horizon is extended.

B Proofs

Proof of proposition 1 (page 14). From equations 5 and 6 and assuming an interior allocation using all available land we find an expression for λ_t

$$\lambda_t = \beta(u'(c_{t+1})(f_2 - f_1) - \lambda_{t+1})$$

Shifting this equation forward by one period gives an expression for λ_{t+1}

$$\lambda_{t+1} = \beta(u'(c_{t+2})(f_2 - f_1) - \lambda_{t+2})$$

Substituting the expression for λ_{t+1} into the expression for λ_t and assuming that there is a two-period cycle, i.e. $x_{1t} = x_{1,t+2}$, $x_{2t} = x_{2,t+2}$, gives

$$\lambda_t = \beta(f_2 - f_1)(u'(c_{t+1}) - \beta u'(c_t)) + \beta^2 \lambda_{t+2}$$

For the two-period cycle to be optimal, this expression for λ_t must be non-negative. We will first find a condition on the allocations in the cycle which guarantees that there exists a sequence of non-negative λ_t 's which satisfy the KKT conditions. This proves sufficiency of the condition. To prove necessity, we will show that for cycles violating this condition, there is no sequence of non-negative λ 's, and therefore no solution to the KKTs.

Assume that $\lambda_t = \lambda_{t+2}$. Applying the assumption and rearranging gives

$$\lambda_t = \frac{\beta(f_2 - f_1)}{1 - \beta^2}(u'(c_{t+1}) - \beta u'(c_t))$$

For this expression to be non-negative we must have

$$u'(c_{t+1}) - \beta u'(c_t) \geq 0 \quad \Rightarrow \quad \frac{u'(c_{t+1})}{u'(c_t)} \geq \beta$$

Similarly for λ_{t+1}

$$u'(c_t) - \beta u'(c_{t+1}) \geq 0 \quad \Rightarrow \quad \frac{u'(c_{t+1})}{u'(c_t)} \leq \frac{1}{\beta}$$

Combining these inequalities

$$\beta \leq \frac{u'(c_{t+1})}{u'(c_t)} \leq \frac{1}{\beta} \tag{25}$$

Therefore, if $\beta \leq \frac{u'(c_{t+1})}{u'(c_t)} \leq \frac{1}{\beta}$, then there exists set of non-negative lambdas, that combined with the cyclical land allocation, solve the solve the two-age-class orchard problem. This proves the sufficiency of equation 25.

□

Proof of corollary 2 (page 15). For each period in a balanced orchard, half of the land is allocated to young trees and half to old trees. Since the land allocation is the same every period, the production is the same every period $c_t = c_{t+1}$.

Applying this production path to inequality 25 gives

$$\beta \leq \frac{u'(c_t)}{u'(c_t)} \leq \frac{1}{\beta} \quad \Rightarrow \quad \beta \leq 1 \leq \frac{1}{\beta}$$

which is true for all $0 < \beta \leq 1$. Hence the balanced orchard is a solution to the two-age-class orchard management problem.

□

Proof of proposition 3 (page 20). Define the function $g(\cdot)$ as

$$\begin{aligned} g(\phi; \beta, f_1, f_2, L) &= \frac{u'(c(\frac{L}{2} - \bar{\phi}))}{u'(c(\frac{L}{2} + \bar{\phi}))} - \beta = 0 \\ &= \frac{u'(\frac{L}{2}(f_1 + f_2) + (f_2 - f_1)\bar{\phi})}{u'(\frac{L}{2}(f_1 + f_2) - (f_2 - f_1)\bar{\phi})} - \beta = 0 \end{aligned}$$

Using the implicit function theorem, the partial derivative of ϕ with respect to any parameter α is

$$\frac{\partial \phi}{\partial \alpha} = -\frac{\frac{\partial g}{\partial \alpha}}{\frac{\partial g}{\partial \phi}}$$

Let $c(x) = f_1 x + f_2 (L - x)$. First, compute the denominator

$$\frac{\partial g}{\partial \phi} = \frac{(f_2 - f_1) [u'(c(\frac{L}{2} + \phi))u''(c(\frac{L}{2} - \phi)) + u'(c(\frac{L}{2} - \phi))u''(c(\frac{L}{2} + \phi))]}{u'(c(\frac{L}{2} - \phi))^2}$$

$\frac{\partial g}{\partial \phi} < 0$ since $u''(\cdot) < 0$. So $\text{sign}(\frac{\partial \phi}{\partial \alpha}) = \text{sign}(\frac{\partial g}{\partial \alpha})$.

Computing the sign of $\frac{\partial \phi}{\partial \beta}$ gives

$$\text{sign}\left(\frac{\partial \phi}{\partial \beta}\right) = \text{sign}(-1) \quad (< 0)$$

Computing $\frac{\partial \phi}{\partial f_1}$ gives

$$\frac{\partial \phi}{\partial f_1} = \frac{(\frac{L}{2} - \phi)u'(c(\frac{L}{2} - \phi))u''(c(\frac{L}{2} + \phi)) - (\frac{L}{2} + \phi)u'(c(\frac{L}{2} + \phi))u''(c(\frac{L}{2} - \phi))}{u'(c(\frac{L}{2} - \phi))^2}$$

$$\text{sign}\left(\frac{\partial \phi}{\partial f_1}\right) > 0 \Leftrightarrow (\frac{L}{2} - \phi)u'(c(\frac{L}{2} - \phi))u''(c(\frac{L}{2} + \phi)) - (\frac{L}{2} + \phi)u'(c(\frac{L}{2} + \phi))u''(c(\frac{L}{2} - \phi)) > 0$$

$$u'(c(\frac{L}{2} - \phi))u''(c(\frac{L}{2} + \phi)) > \frac{(\frac{L}{2} + \phi)}{(\frac{L}{2} - \phi)} u'(c(\frac{L}{2} + \phi))u''(c(\frac{L}{2} - \phi))$$

$$\frac{u'(c(\frac{L}{2} - \phi))u''(c(\frac{L}{2} + \phi))}{u'(c(\frac{L}{2} + \phi))u''(c(\frac{L}{2} - \phi))} < \frac{(\frac{L}{2} + \phi)}{(\frac{L}{2} - \phi)} \quad (\text{Since } u''(\cdot) < 0)$$

$$\frac{A(c(\frac{L}{2} + \phi))}{A(c(\frac{L}{2} - \phi))} < \frac{(\frac{L}{2} + \phi)}{(\frac{L}{2} - \phi)}$$

Where $A(x) = \frac{-u''(x)}{u'(x)}$, the Arrow-Pratt measure of absolute risk aversion.

Computing $\frac{\partial \phi}{\partial f_2}$ gives

$$\frac{\partial \phi}{\partial f_2} = \frac{(\frac{L}{2} + \phi)u'(c(\frac{L}{2} - \phi))u''(c(\frac{L}{2} + \phi)) - (\frac{L}{2} - \phi)u'(c(\frac{L}{2} + \phi))u''(c(\frac{L}{2} - \phi))}{u'(c(\frac{L}{2} - \phi))^2}$$

$$\text{sign} \left(\frac{\partial \phi}{\partial f_2} \right) > 0 \Leftrightarrow (\frac{L}{2} + \phi)u'(c(\frac{L}{2} - \phi))u''(c(\frac{L}{2} + \phi)) - (\frac{L}{2} - \phi)u'(c(\frac{L}{2} + \phi))u''(c(\frac{L}{2} - \phi)) > 0$$

$$u'(c(\frac{L}{2} - \phi))u''(c(\frac{L}{2} + \phi)) > \frac{(\frac{L}{2} - \phi)}{(\frac{L}{2} + \phi)} u'(c(\frac{L}{2} + \phi))u''(c(\frac{L}{2} - \phi))$$

$$\frac{u'(c(\frac{L}{2} - \phi))u''(c(\frac{L}{2} + \phi))}{u'(c(\frac{L}{2} + \phi))u''(c(\frac{L}{2} - \phi))} < \frac{(\frac{L}{2} - \phi)}{(\frac{L}{2} + \phi)} \quad (\text{Since } u''(\cdot) < 0)$$

$$\frac{A(c(\frac{L}{2} + \phi))}{A(c(\frac{L}{2} - \phi))} < \frac{(\frac{L}{2} - \phi)}{(\frac{L}{2} + \phi)}$$

Where $A(x) = \frac{-u''(x)}{u'(x)}$, the Arrow-Pratt measure of absolute risk aversion.

Computing $\frac{\partial \phi}{\partial L}$ gives

$$\frac{\partial \phi}{\partial L} = \frac{(f_1 + f_2) (u'(c(\frac{L}{2} + \phi))u''(c(\frac{L}{2} - \phi)) - u'(c(\frac{L}{2} - \phi))u''(c(\frac{L}{2} + \phi)))}{2u'(c(\frac{L}{2} + \phi))^2}$$

$$\text{sign} \left(\frac{\partial \phi}{\partial L} \right) > 0 \Leftrightarrow u'(c(\frac{L}{2} + \phi))u''(c(\frac{L}{2} - \phi)) - u'(c(\frac{L}{2} - \phi))u''(c(\frac{L}{2} + \phi)) > 0$$

$$u'(c(\frac{L}{2} + \phi))u''(c(\frac{L}{2} - \phi)) > u'(c(\frac{L}{2} - \phi))u''(c(\frac{L}{2} + \phi))$$

$$\frac{u'(c(\frac{L}{2} + \phi))u''(c(\frac{L}{2} - \phi))}{u'(c(\frac{L}{2} - \phi))u''(c(\frac{L}{2} + \phi))} < 1 \quad (\text{Since } u''(\cdot) < 0)$$

$$\frac{A(c(\frac{L}{2} + \phi))}{A(c(\frac{L}{2} - \phi))} > 1$$

□

Proof of proposition 4 (page 21). Define ϕ , the maximum radius of the cycle region, as in equation

11. Assume that the land constraint binds every period, so we can define a land allocation as $\mathbf{x}_t = (L - x_t, x_t)$, where x_t is the area of land allocated to old trees in period t . Let $x_{t+1} = P(x_t)$ be the function that returns the optimal area of old trees in period $t + 1$ given the area of old trees in period t . Assume that it exists and is continuous.

We will consider the optimal transition for three regions in $x_t \in [0, L]$. Let region one be $[0, \frac{L}{2} - \phi)$, region two be $[\frac{L}{2} - \phi, \frac{L}{2} + \phi]$, and region three be $(\frac{L}{2} + \phi, L]$. We will construct the optimal transition function piecewise across these three regions

$$P(x_t) = \begin{cases} P_1(x_t) & \text{for } x_t \in [0, \frac{L}{2} - \phi) \\ P_2(x_t) & \text{for } x_t \in [\frac{L}{2} - \phi, \frac{L}{2} + \phi] \\ P_3(x_t) & \text{for } x_t \in (\frac{L}{2} + \phi, L] \end{cases}$$

We know from proposition 1 that $P_2(x_t) = L - x_t$. We will begin by showing that $P_3(x_t) = L - x_t$ as well, and then use this result to show that $P_1(x_t) = \frac{L}{2} - \phi$, thus finding the piecewise definition of $P(x_t)$.

The optimal transition function will depend on the optimal value of the aging constraint multiplier, λ_t . If $\lambda_t > 0$, then the aging constraint binds, and $x_{t+1} = L - x_t$. On the other hand, if $\lambda_t = 0$, then $x_{t+1} \leq L - x_t$ and we will need to pin down its value.

We will show that for x_t in region 3, starting with $\lambda_t = 0$ implies there is no solution to the KKTs in period $t + 1$. Therefore $\lambda_t > 0$ for x_t in region 3.

Recall the Euler equation

$$\lambda_t = \beta(u'(c_{t+1})f_2 - u'(c_{t+1})f_1 - \lambda_{t+1})$$

Assume $x_t \in (\frac{L}{2} + \phi, L]$ and $\lambda_t = 0$. Therefore $\lambda_{t+1} = \beta u'(c_{t+1})(f_2 - f_1) > 0$. Iterating the Euler equation and using this value for λ_{t+1} gives an expression for λ_{t+2}

$$\lambda_{t+2} = (f_2 - f_1) \left(u'(c_{t+2}) - \frac{1}{\beta} u'(c_{t+1}) \right)$$

We must have $\lambda_{t+2} \geq 0$ to satisfy the KKT conditions.

$$\begin{aligned}
& \lambda_{t+2} \geq 0 \\
\Rightarrow & u'(c_{t+2}) \geq \frac{1}{\beta} u'(c_{t+1}) \\
\Rightarrow & x_{t+2} < x_{t+1} < x_t \quad (\text{with strict inequality because } \beta < 1)
\end{aligned}$$

This is incompatible with $\lambda_t = 0$ and $\lambda_{t+1} > 0$, which imply that $x_{t+2} = L - x_{t+1} \geq x_t$. Therefore $x_t \in (\frac{L}{2} + \phi, L]$ implies $\lambda_t > 0$, and $P_3(x_t) = L - x_t$.

We now turn to characterizing $P_1(x_t)$. For x_t in region 1, we begin by showing that assuming $\lambda_t > 0$ implies there is no solution to the KKTs in period $t+1$. We then find the optimal transition for $x_{t+1} \leq L - x_t$ given $\lambda_t = 0$.

For x_t in region 1, if $\lambda_t > 0$, then $x_{t+1} = L - x_t$, which is in region 3. From our previous result, $x_{t+2} = P_3(x_{t+1}) = x_t$, which implies a cycle. However, by proposition 1, the only cycles consistent with the KKTs are those with $x_t, x_{t+1} \in [\frac{L}{2} - \phi, \frac{L}{2} + \phi]$. Therefore, $\lambda_t = 0$ in region 1.

Finally, for x_t in region 1, let $\lambda_t = 0$, which implies $x_{t+1} \leq L - x_t \in (\text{region 3})$, so $\lambda_{t+1} > 0$. Further, from before, x_{t+1} in region 3, implies that $\lambda_{t+2} = 0$. Using the Euler equation and iterating by one period we get an expression for λ_{t+2} :

$$\begin{aligned}
& (f_2 - f_1) \left(u'(c_{t+2}) - \frac{1}{\beta} u'(c_{t+1}) \right) = 0 \\
\Rightarrow & \frac{u'(c_{t+1})}{u'(c_{t+2})} = \beta \\
\Rightarrow & c_{t+1} = (f_1(\frac{L}{2} - \phi) + f_2(\frac{L}{2} + \phi)) \\
\Rightarrow & x_{t+1} = \frac{L}{2} + \phi
\end{aligned}$$

So the only allocation of old trees in period $t+1$ consistent with the KKTs when x_t is in region 1 is $x_{t+1} = \frac{L}{2} + \phi$. Hence $P_1(x_t) = \frac{L}{2} + \phi$, a constant. The land allocated to old trees in period $t+1$ is independent of the allocation of old trees in period t , so long as it is in region 1. Any allocation in region 1 moves to the upper boundary of region 2 in the next period, and then remains in region 2 thenceforth.

The optimal transition rule is thus

$$P(x_t) = \begin{cases} \frac{L}{2} + \phi & \text{for } x_t \in [0, \frac{L}{2} - \phi) \\ L - x_t & \text{for } x_t \in [\frac{L}{2} - \phi, \frac{L}{2} + \phi] \\ L - x_t & \text{for } x_t \in (\frac{L}{2} + \phi, L] \end{cases}$$

as shown in figure 7.

□